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THE DYNAMICS AND CONTROL OF AXIAL SATELLITE GYGOSTAS OF VARIABLE STUCTURE

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This paper presents the study of the dynamics and control of an axial variable structure satellite (asymmetric platform and an axisymmetric rotor). Inertia moments of the rotor change slowly over time. The dynamics of the satellite is described by using ordinary differential equations with Serret-Andoyer canonical variables. For undisturbed motion, the stationary solutions are found, and their stability is studied. The control law is obtained for the satellite with variable structure on the basis of the stationary solutions. By means of computer numerical simulations, we have shown that the motion of the satellite controlled by founded internal torque is stable.

INTRODUCTION

Artificial satellites can contain one or more spinning rotors to provide gyroscopic stability of a desired orientation of the vehicle. Dual-spin satellites use the spin of a rotor to maintain pointing accuracy of an antenna platform or a solar sail. Some types of satellites, on the other hand, use small but rapidly spinning momentum wheels to control the attitude of a large platform. In this paper we consider rotational motion of axial dual-spin spacecrafts without external torque. In recent years, the gyrostat model has attracted the attention of aerospace engineers (see, for instance References 1-17).

In this paper we deal with a particular case of gyrostat; it consists of a platform *P* with triaxial ellipsoid of inertia and a symmetrical rotor *R* whose axis of symmetry is aligned along one of the principal axes of inertia of the body. The attitude motion of the gyrostat is described by a system of three differential equations that determine the position of the angular velocity with respect to the body frame (Euler's equations), a single equation to describe the motion of the rotor relative to the platform.

This paper presents the study of the dynamics and control of an axial satellite gyrostat with variable structure and free of external torques. Depending on the relationship of inertia moments the paper discusses three basic types of gyrostats: oblate, prolate and intermediate and the two boundary types: oblate-intermediate, prolate-intermediate. During the motion of a satellite the inertia moments of the rotor change slowly in time, which may be related to the deployment of solar panels, solar sails and other constructions. In this case, the satellite gyrostat can take place all the types from prolate to oblate or vice versa. An internal control torque can be used to

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provide a stable motion of the variable structure satellite gyrostat. Therefore, we assume that the angular momentum of the rotor about its axis of symmetry isn't constant.

The purpose of this paper is to find the control for the satellite gyrostat. The control law should allow keeping a stable motion in the vicinity of the equilibrium position for slowly changing the inertia moments of the rotor in time. The main idea of the stabilization method is conservation of the stable position by selecting the the internal torque. We solve the following tasks to achieve this purpose. The dynamics of the satellite is described by using ordinary differential equations with Serret-Andoyer[18,19] canonical variables. The equations of motion have a simple dimensionless form and contain a small parameter[12]. For undisturbed motion (the small parameter equal to zero), when the inertia moments of the satellite gyrostat aren't changed and the internal torque is equal to zero, the stationary solutions are found, and their stability is studied for the all the types of the gyrostats. For disturbed motion of the gyrostat with variable structure the control law obtained on the basis of the stationary solutions.

EQUATIONS OF MOTION

The differential equations of the motion for the angular momentum variables of a rigid axial gyrostat with zero external torque may be written as[8]

$$\frac{dh_1}{dt} = \frac{I_2 - I_3}{I_2 I_3} h_2 h_3, \tag{1}$$

$$\frac{dh_2}{dt} = \left(\frac{I_3 - I_P}{I_3} h_1 - h_a \right) \frac{h_3}{I_P}, \tag{2}$$

$$\frac{dh_3}{dt} = \left(\frac{I_P - I_2}{I_2} h_1 + h_a \right) \frac{h_2}{I_P}, \tag{3}$$

$$\frac{dh_a}{dt} = \tilde{g}_a \tag{4}$$

where e_i are principal axes of $P+R$ ($i=1,2,3$); $\tilde{g}_a$ is the torque applied by P on R about e_1; $h_a=I_S(\omega_S+\omega_1)$ is the angular momentum of R about e_1; $h_1=I_1\omega_1+I_s\omega_s$ is the angular momentum of $P+R$ about e_1; $h_i = I_i\omega_i$ are the angular momentum of $P+R$ about e_i ($i=2,3$); I_i are the moments of inertia of $P+R$ about e_i ($i=1,2,3$); $I_P=I_1 - I_S$ is the moment of inertia of P about e_1; I_s is the moment of inertia of R about e_1; t is time, ω_i are the angular velocities of P about e_i ($i=1,2,3$); ω_s – is the angular velocity of R about e_1 relative to P.

Since there is no external torque, then the angular momentum is conserved and the first integral of the motion is

$$G = \sqrt{h_1^2 + h_2^2 + h_3^2} = const. \tag{5}$$

The equations of the motion can be simplified by using two canonical Serret-Andoyer variables[18,19]: l, L (Figure 1). Using the change of variables

$$h_1 = L, \quad h_2 = \sqrt{G^2 - L^2}\sin l, \quad h_3 = \sqrt{G^2 - L^2}\cos l \tag{6}$$

we obtain the equations of the motion in terms of the Serret-Andoyer variables

$$\frac{dl}{dt}=\frac{1}{I_P}\left[L-h_a-\frac{1}{2}L\ \ a+b+(b-a)\cos 2l\ \ \right], \tag{7}$$

$$\frac{dL}{dt}=\frac{1}{2I_P}(b-a)\ \ G^2-L^2\ \ \sin 2l\,, \tag{8}$$

$$\frac{dh_a}{dt}=g_a \tag{9}$$

where

$$a=\frac{I_P}{I_2},\ \ b=\frac{I_P}{I_3}.$$

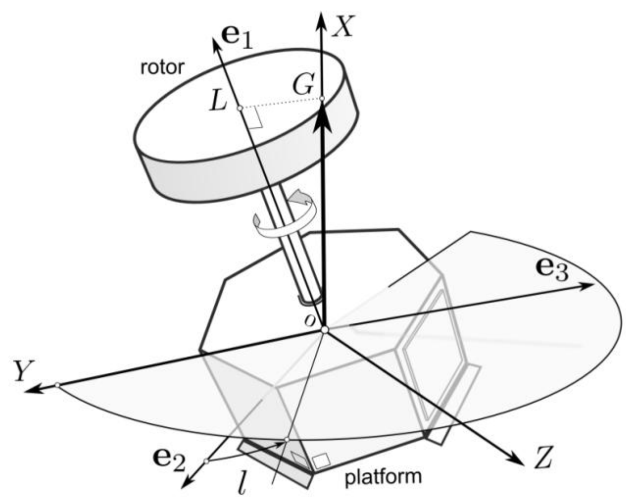

Fig. 1 The axial gyrostat

Assume moreover that

$$I_2>I_3,\ \ b>a\ \ .$$

The transformation of the Equations (7)-(9) to a dimensionless form is obtained by scaling two momentum, time and axial torque, as follows:

$$s=\frac{L}{G},\ d=\frac{h_a}{G}\ ,\ \tau=t\frac{G}{I_p},\ g_a=\frac{\tilde{g}_a I_p}{G^2}. \tag{10}$$

The change of variables (Eq. (10)) leads to the equivalent set of dimensionless equations:

$$l'=s-d-\frac{s}{2}\left[\ a+b\ \ +(b-a)\cos 2l\right], \tag{11}$$

$$s' = \frac{1}{2}(b-a)\left(1-s^2\right)\sin 2l, \tag{12}$$

$$d' = g_a \tag{13}$$

where $(.)' = d(.)/d\tau$.

Let us assume the inertia moments of the axisymmetric rotor R about e_1, e_2, e_3 are continuous functions of the dimensionless time

$$I_S = I_S(\tau), I_R = I_R(\tau) \tag{14}$$

For this case, we carried out a separate study and showed that the form of the motion Equations (11)-(13) of the gyrostat doesn't change. Furthermore, we assume the derivative of the moment of inertia of the rotor and the internal point by small

$$\frac{dI_S}{d\tau}, \frac{dI_R}{d\tau}, g_a = O(\varepsilon) \tag{15}$$

where ε is a small parameter.

UNPERTURBED MOTION AND STATIONARITY SOLUTIONS

At ε=0 the perturbed Equations (11)-(13) are reduced to an unperturbed canonical system with one degree of freedom

$$l' = \frac{\partial H}{\partial s} = s - d - \frac{s}{2}\left[(a+b) + (b-a)\cos 2l\right], \tag{16}$$

$$s' = -\frac{\partial H}{\partial l} = \frac{1}{2}(b-a)\left(1-s^2\right)\sin 2l \tag{17}$$

where $a, b, d = const$; H is Hamiltonian by

$$H(l,s) = \frac{1-s^2}{4}\left[(a+b) + (b-a)\cos 2l\right] + \frac{s^2}{2} - sd = h = const. \tag{18}$$

Solving the Eq. (16) with respect to $\cos 2l$, we get an equation of the phase trajectory:

$$\cos 2l = \frac{(a+b-2)s^2 + 4ds + 4h - a - b}{\left(1-s^2\right)(b-a)} \tag{19}$$

Canonical Eq. (16) and Eq. (17) have four stationary solutions[12]:

$$\cos(2l_*) = 1, \qquad s_* = d/(1-b), \tag{20}$$

$$\cos(2l_*) = -1, \qquad s_* = d/(1-a), \tag{21}$$

$$\cos(2l_*) = (2-a-b-2d)/(b-a), \qquad s_* = 1, \tag{22}$$

$$\cos(2l_*) = (2-a-b+2d)/(b-a), \qquad s_* = -1. \tag{23}$$

Aslanov proved[12] that the first stationary solution Eq. (20) is stable if $b>1$ $I_P>I_3$ and unstable if $b<1$ $I_P<I_3$, the second stationary solution Eq. (21) is stable if $a<1$ $I_P<I_2$ and unstable if $a>1$ $I_P>I_2$, the third and fourth stationary solutions (Eq. (22) and (23)) are unstable.

Let us give complete classification of all types gyrostats depending on the ratio of moments of inertia:

1) Oblate Gyrostat: $I_p>I_2>I_3$ $b>a>1$,

2) Oblate-Intermediate Gyrostat: $I_p=I_2>I_3$ $b>a=1$,

3) Intermediate Gyrostat: $I_2>I_p>I_3$ $b>1>a$,

4) Prolate-Intermediate Gyrostat: $I_2>I_p=I_3$ $b=1>a$,

5) Prolate Gyrostat: $I_2>I_3>I_p$ $1>b>a$.

Gyrostats 1), 3) and 5) correspond to areas with the same numbers in Figure 2, gyrostat 2) corresponds to the border between areas 1 and 3 and type 4) – to the border between areas 3 and 5.

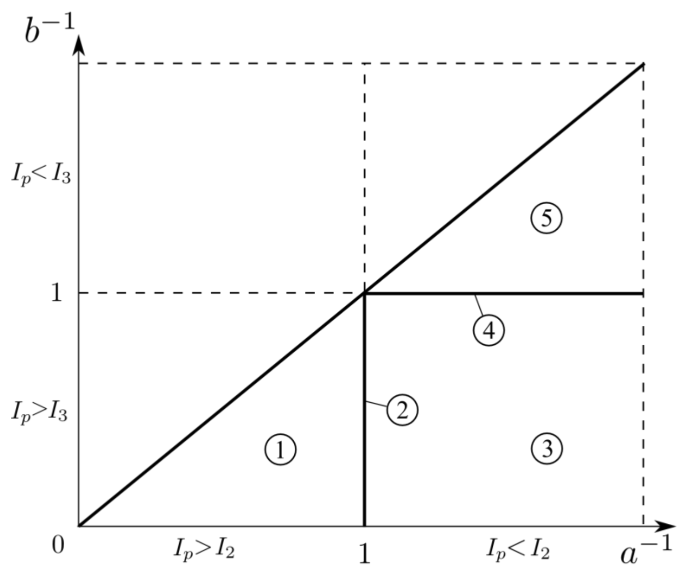

FIGURE 2.Partition of the parameter plane b^{-1}, a^{-1}

The coordinates critical points for all types of the gyrostats given in Table 1, where the subscripts “c” and “s” denote centers and saddles, respectively. The coordinates of the critical points correspond to stationary solutions obtained above Eq. (13)-(16).

Examples of phase trajectories for the Oblate Gyrostat (1a), the Prolate Gyrostat (5b) and the Intermediate Gyrostat (3b) are shown in s, l coordinates in Figure 3-5. In Figure 5, there are two types of separatrix for the Intermediate Gyrostat (3b), one of which contains saddles with s_s=-sgn d and another saddle width s_s=sgn d . In the phase space bounded by these separatrixes, there is continuous motion with sequential change in the sign of the momentum s.

Table 1. The critical points coordinate for various types of gyrostats

Case	Gyrostat type	The kinematic conditions	The critical points (l, s): Center	The critical points (l, s): Saddle
1a	Oblate $b > a > 1$	$\left\|\frac{d}{1-a}\right\| \le 1$	$l_c = 0$, $s_c = d/(1-b)$	$l_s = \frac{\pi}{2}$, $s_s = d/(1-a)$
1b		$\left\|\frac{d}{1-a}\right\| > 1$	$l_c = 0$, $s_c = d/(1-b)$	$\cos 2l_s = \frac{2-a-b+2d}{b-a}$, $s_s = -\operatorname{sgn} d$
2	Oblate-Intermediate $b > a = 1$	$\left\|\frac{d}{1-a}\right\| \to \infty$	$l_c = 0$, $s_c = d/(1-b)$	$\cos 2l_s = -1 + \frac{2d}{b-1}$, $s_s = -\operatorname{sgn} d$
3a	Intermediate $b > 1 > a$	$\left\|\frac{d}{1-a}\right\| > 1$	$l_c = 0$, $s_c = d/(1-b)$	$\cos 2l_s = \frac{2-a-b+2d}{b-a}$, $s_s = -\operatorname{sgn} d$
3b		$\left\|\frac{d}{1-a}\right\| \le 1$, $\left\|\frac{d}{1-b}\right\| \le 1$	$l_c = 0$, $s_c = d/(1-b)$	$\cos 2l_s = \frac{2-a-b+2d}{b-a}$, $s_s = -\operatorname{sgn} d$
			$l_c = \pm\frac{\pi}{2}$, $s_c = d/(1-a)$	$\cos 2l_s = \frac{2-a-b-2d}{b-a}$, $s_s = \operatorname{sgn} d$
3c		$\left\|\frac{d}{1-b}\right\| > 1$	$l_c = \pm\frac{\pi}{2}$, $s_c = d/(1-a)$	$\cos 2l_s = \frac{2-a-b-2d}{b-a}$, $s_s = \operatorname{sgn} d$
4	Problate-Intermediate $b = 1 > a$	$\left\|\frac{d}{1-b}\right\| \to \infty$	$l_c = \pm\frac{\pi}{2}$, $s_c = d/(1-a)$	$\cos 2l_s = 1 - \frac{2d}{1-a}$, $s_s = \operatorname{sgn} d$
5a	Prolate $1 > b > a$	$\left\|\frac{d}{1-b}\right\| > 1$	$l_c = \pm\frac{\pi}{2}$, $s_c = d/(1-a)$	$\cos 2l_s = \frac{2-a-b-2d}{b-a}$, $s_s = \operatorname{sgn} d$
5b		$\left\|\frac{d}{1-b}\right\| \le 1$	$l_c = \pm\frac{\pi}{2}$, $s_c = d/(1-a)$	$l_s = 0$, $s_s = d/(1-b)$

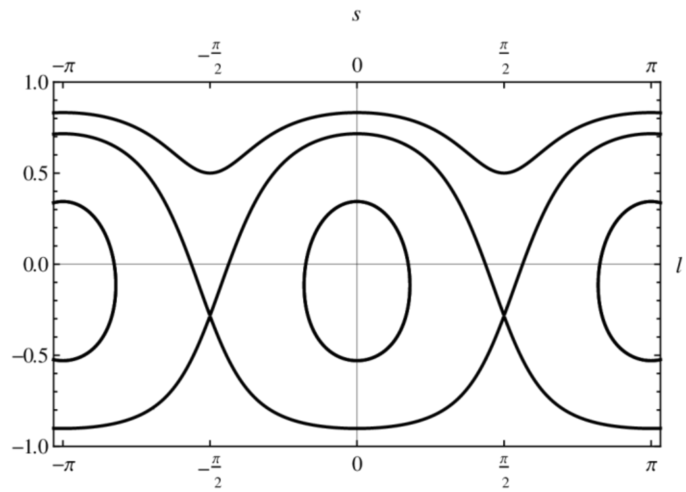

FIGURE 3. Phase trajectories for the Oblate Gyrostat (1a): I_2=0.85 kgm^2, I_3=0.65 kgm^2, I_p=1.0 kgm^2, d=0.05 [s_c=-0.093 (l_c=0), s_s=-0.283 (l_s=±π/2)]

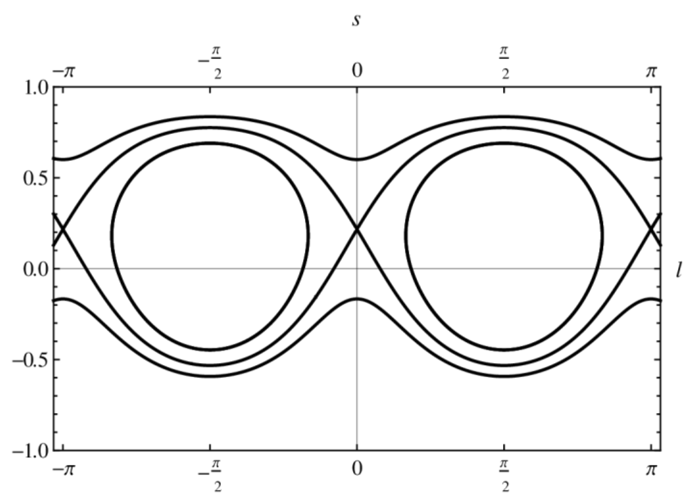

FIGURE 4 . Phase trajectories for the Prolate Gyrostat (5b): I_2=0.85 kgm^2, I_3=0.65 kgm^2, I_p=0.5 kgm^2, d=0.05 [s_c=0.2125 (l_c=±π/2), s_s=0.2167 (l_s=0)]

GYROSTAT STABILIZATION

Due to change of gyrostat's moments of inertia the phase space is deforming and uncontrolled gyrostat can change its motion type and lose its axis orientation. Here we set up the problem to keep gyrostat initial state in phase space. The main idea of proposed stabilization method is conservation of the stable position by selecting the internal torque. It's enough to differentiate with respect to the dimensionless time stationary solutions Eq. (20), (21). For the oblate gyrostat ($b>a>1$) stationary point in the phase space is defined as

$$s_c = s_* = \frac{d}{1-b} \tag{24}$$

than

$$d = s_*(1-b) \tag{25}$$

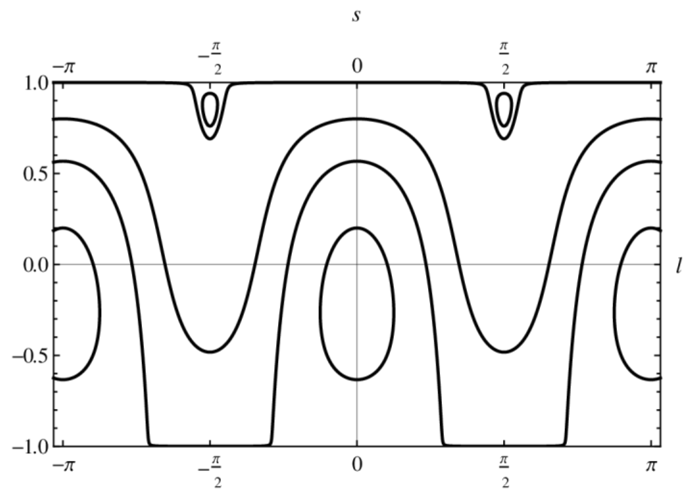

Figure 5. Phase trajectories for the Intermediate Gyrostat (3b): I_2=0.85 kgm^2, I_3=0.65 kgm^2, I_p=0.8 kgm^2, d=0.05. Centers: [s_c=-0.2167 (l_c=0), s_c=0.85 (l_c=±π/2)]. Saddles: [s_s=1.0 (l_s=±1.3953), s_s=-1 (l_s=±0.9109)].

Differentiating Eq. (25) we get control law

$$d' = g_a = s_*(1-b)' \tag{26}$$

or in dimension parameters

$$d' = g_a = \frac{I_P I_R'}{I_3^2} s_* \tag{27}$$

By a similar way for the prolate gyrostat we get

$$s_c = s_* = \frac{d}{1-a} \tag{28}$$

than

$$d' = g_a = \frac{I_P I_R'}{I_2^2} s_* \tag{29}$$

NUMERICAL RESULTS

Here we compare behaviors of the two gyrostats: the uncontrolled gyrostat and gyrostat controlled by internal torque. Let us suppose that the gyrostat's moments of inertia change in following way:

$$I_R(\tau) = I_{R0} - k_2\tau\ ,\ I_S(\tau) = k_S I_R(\tau) \tag{30}$$

Values of the gyrostat's parameter are given in Table 2. There I_{Pi} – are platform's moments of inertia about e_i ($i=1,2,3$).

Table 2. Values of the parameters employed for the numerical simulation

Parameter	Value	Parameter	Value
$I_{P1}=I_P$	1	G	1
I_{P2}	0.8	k_2	$4.375\cdot 10^{-4}$
I_{P3}	0.7	k_s	0.3
I_{R0}	0.357		

Uncontrolled gyrostat

There are two initial conditions are considered. At first we consider the behavior of the uncontrolled gyrostat (g_a=0) with initial conditions corresponded to the stationary point of the prolate gyrostat near $s=1$:

$$s_0 = 0.99,\ l_0 = \pi / 2,\ d_0 = s_0(1-a) = 0.1343 \tag{31}$$

In the other words gyrostat starts its motion from the stationary point and gyrostat's with axis e_1 oriented close to angular momentum vector.

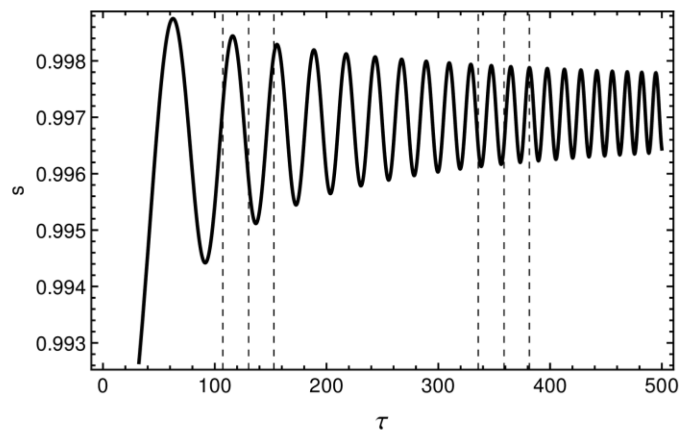

Figure 6. The state variable s

Figure 6 shows that in this case gyrostat continues its motion near the initial orientation in the sense that $s \approx s_0$. We note that changes in moments of inertia affect the angular velocities of R about e_1 relative to P (Figure 7). Angular velocities of P about e_i ($i=1,2,3$) are oscillating with high frequency (Figure 8) that caused additional acceleration of the gyrostat spacecraft.

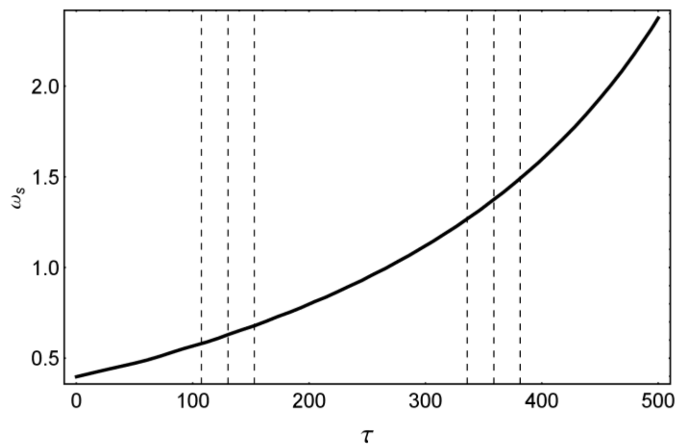

Figure 7. The angular velocities of R about e_1 relative to P

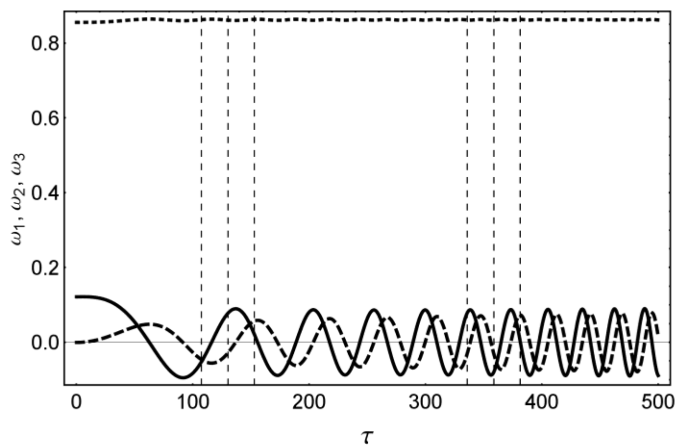

Figure 8. The angular velocities of *P* (ω_1 – dotted, ω_2 – solid, ω_3 – dashed)

Next we consider case when gyrostat starts its motion from stationary point at $s_0=0.5$:

$$s_0 = 0.5,\ l_0 = \pi / 2,\ d_0 = \mathrm{s}_0(1-a) = 0.06785 \tag{32}$$

In this case gyrostat loses its orientation (Figure 9). The angle between e_1 axis and angular momentum vector (this angle can be found as *arccos s*) changes sufficiently. As we see in the first case changes in moments of inertia affect the angular velocities of *R* about e_1 relative to *P* (Figure 10) and also angular velocities of *P* about e_i ($i=1,2,3$) are changing with high frequency (Figure 11).

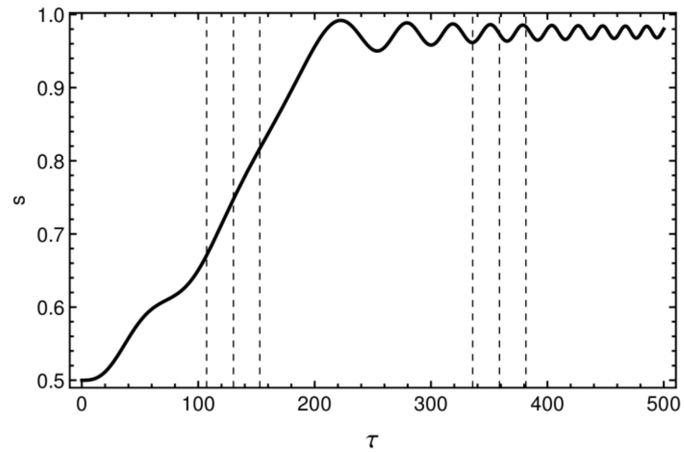

Figure 9. The state variable s

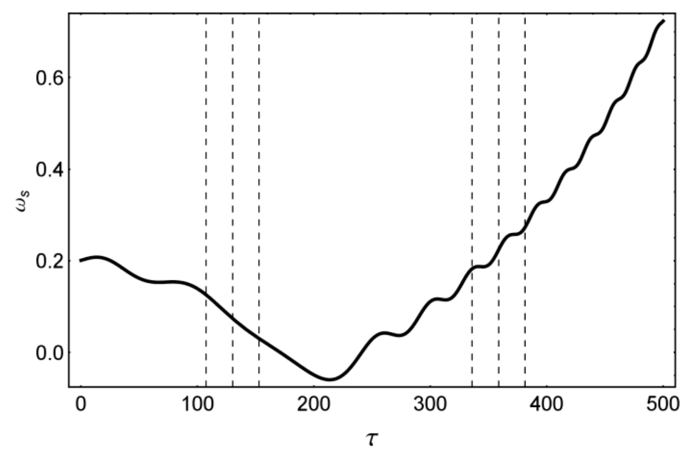

Figure 10. The angular velocities of R about e_1 relative to P

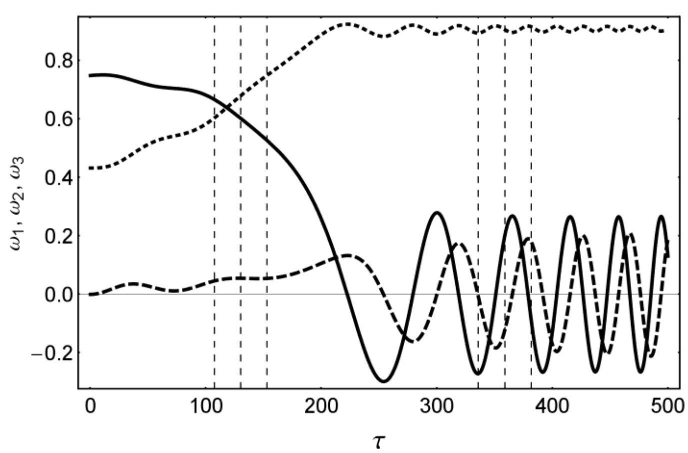

Figure 11. The angular velocities of *P* (ω_1 – dotted, ω_2 – solid, ω_3 – dashed)

Controlled gyrostat

Here we suppose that internal torque g_a defined by Eq. (27) acts between P and R. Internal moment is defined by (29) because of gyrostat's initial state. At first let us consider case when initial conditions are defined by Eq. (31). In this case *s* remains practically constant. The angular velocities of *R* about e_1 relative to *P* is monotonically decreased due to internal torque (Figure 12). The angular velocities of *P* about e_i ($i=1,2,3$) are monotonic functions of τ (Figure 13) and we can expect that acceleration due to attitude motion of the internal torque controlled gyrostat will be less than acceleration of the uncontrolled gyrostat.

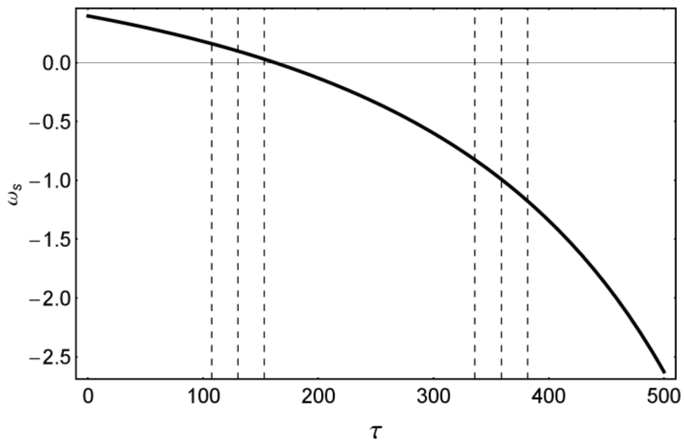

Figure 12. The angular velocities of R about e_1 relative to P

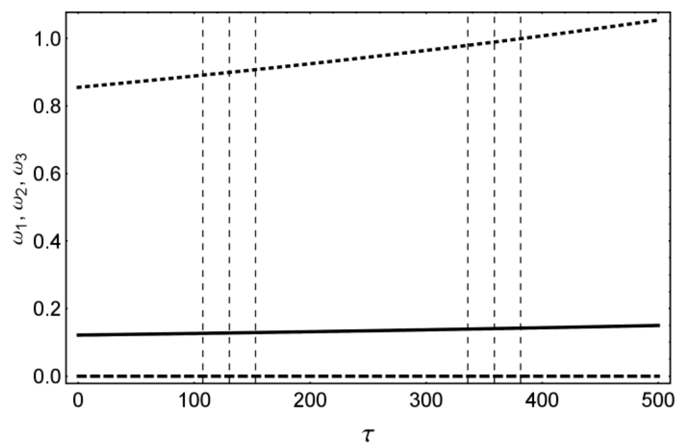

Figure 13. The angular velocities of *P* (ω_1 – dotted, ω_2 – solid, ω_3 – dashed)

Next we consider case when gyrostat starts with initial conditions are defined by Eq. (32). As in the previous case, *s* remains practically constant and the angular velocities of *R* about e_1 relative to *P* is decreased (Figure 14). The angular velocities of *P* about e_i ($i=1,2,3$) also are monotonic functions of τ (Figure 15).

We can see that system preserves it's state in phase space despite to deformation during changing of the moments of inertia of the rotor. Note that internal torque is obtained for stable point

$$s_c = \frac{d_0}{1-b}, l_c = \frac{\pi}{2} \tag{33}$$

i.e. for the prolate gyrostat. Therefore control Eq. (29) is correct while *a<1*. When *a>1* gyrostat becomes oblate and (33) defines an unstable saddle point.

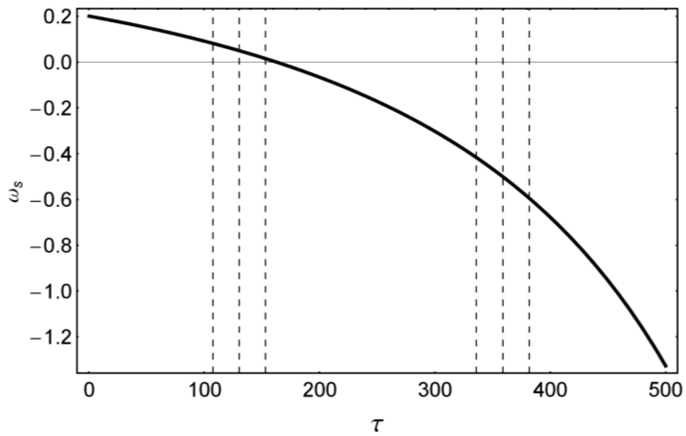

Figure 14. The angular velocities of R about e_1 relative to P

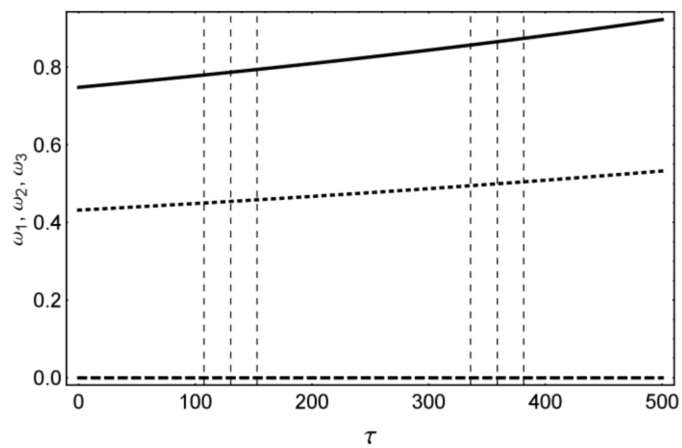

Figure 15. The angular velocities of *P* (ω_1 – dotted, ω_2 – solid, ω_3 – dashed)

CONCLUSION

The dynamics of the dual-spin gyrostat spacecraft is described by using ordinary differential equations with Serret-Andoyer canonical variables. The equations of motion have a simple dimensionless form and contain a small parameter. For undisturbed motion the stationary solutions are found, and their stability is studied for the all the types of the gyrostats. For disturbed motion of the gyrostat with variable structure the control law obtained on the basis of the stationary solutions. It's shown that uncontrolled gyrostat satellite can lose its axis orientation because of change in moments of inertia of the rotor. The oscillations of the angular velocities and accelerations of the gyrostat accompany changes in the inertia moments of the rotor. Obtained internal torque keeps axis orientation of the gyrostat and get angular velocities and accelerations monotonic functions of time. Several numerical examples are given to confirm effectiveness of the control.

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